On super antimagic total labeling of Harary graph

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Abstract. This paper deals with two types of graph labelings namely, the super \((a, d)\)-edge antimagic total labeling and super \((a, d)\)-vertex antimagic total labeling on the Harary graph \(C_n\). We also construct the super edge-antimagic and super vertex-antimagic total labelings for a disjoint union of \(k\) identical copies of the Harary graph.

Keywords : Super antimagic total labeling, Harary graph.

1 Introduction

All graphs in this paper are finite, simple and undirected. The graph \(G\) has the vertex-set \(V(G)\) and edge-set \(E(G)\). A general reference for graph-theoretic ideas can be seen in [10].

A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper the domain will be the set of all vertices and edges and such a labeling is called a total labeling. Some labelings use the vertex-set only, or the edge-set only, and we shall call them vertex-labelings and edge-labelings respectively. Other domains are possible. The most complete recent survey of graph labelings can be seen in [6]. There are many types of graph labelings, for example harmonious, cordial, graceful and antimagic. In this paper, we focus on two types of labelings called super \((a, d)\)-edge antimagic total labeling and super \((a, d)\)-vertex antimagic total labeling. A graph \(G\) is called \((a, d)\)-edge antimagic total \(((a, d)\text{-EAT})\) if there exist integers \(a > 0, d \geq 0\) and a bijection \(\lambda : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\} \) such that \(W = \{w(xy) : xy \in E\}\) forms an arithmetic progression starting from \(a\) with the
difference \( d \), where \( w(xy) = \lambda(x) + \lambda(y) + \lambda(xy) \). \( W \) is called the set of edge-weights of the graph \( G \). Additionally, if \( \lambda(V) = \{1, 2, ..., |V|\} \) then \( G \) is super \((a, d)\)-EAT. Similarly, a graph \( G \) is called \((a, d)\)-vertex antimagic total \((a, d)\)-VAT) if there exist integers \( a > 0, d \geq 0 \) and a bijection \( \lambda : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\} \) such that \( W = \{w(x) : x \in V\} \) forms an arithmetic progression starting from \( a \) with the difference \( d \), where \( w(x) = \lambda(x) + \sum_{xy \in E(G)} \lambda(xy) \) where the sum is taken over all vertices \( y \) which are adjacent to \( x \). In this case, \( W \) is called the set of vertex-weights of the graph \( G \). In particular, an \((a, d)\)-VAT labeling \( \lambda \) of graph \( G \) is super if \( \lambda(V) = \{1, 2, ..., |V|\} \).

A number of classification studies on super \((a, d)\)-EAT labeling (resp. \((a, d)\)-EAT) for connected graphs has been extensively investigated. For instances, in [3], Baca et al. showed that a wheel \( W_n \) has a super \((a, d)\) edge-EAT labeling if and only if \( d = 1 \) and \( n \equiv 1 \text{(mod 4)} \). A.A.G. Ngurah and E.T. Baskoro in [5] proved that for every Petersen graph \( P(n, m), n \geq 3, 1 \leq m \leq \frac{n^2}{2} \), has a super \((4n + 2, 1)\)-EAT labeling.

Given an \((a, d)\)-EAT labeling \( \lambda \) on a graph \( G \) with \( p \) vertices and \( q \) edges. Then its dual labeling \( \lambda' \) can be defined [3] by
\[
\lambda'(x) = p + q + 1 - \lambda(x), \text{ for any vertex } x, \text{ and}
\lambda'(xy) = p + q + 1 - \lambda(xy), \text{ for any edge } xy.
\]
Thus, we have the following theorem.

**Theorem A.** [8] Let \( G \) be a graph with \( p \) vertices and \( q \) edges. If \( G \) has an \((a, d)\)-EAT labeling then \( G \) has an \((3p+3q+3-a-(q-1)d, d)\)-EAT labeling as its dual. \( \square \)

For \( t \geq 2 \) and \( n \geq 4 \), a Harary graph \( C_n^t \) is a graph constructed from a cycle \( C_n \) by joining any two vertices at distance \( t \) in \( C_n \). Figure 1 shows the Harary graph \( C_7^2 \).

![Harary graph C_7^2](image-url)
Super \((a, d)\)-EAT labeling on Harary graph has been studied by Baca et. al \([1]\) and Murugan \([4]\). Some of their results are presented in the following three theorems.

**Theorem B.**
For odd \(n, n \geq 5\), and for all possible values of \(t\), the graph \(C_n^t\) has a super \((2n + 2, 1)\)-EAT labeling.

**Theorem C.**
For even \(n, n \geq 6\), and for odd \(t, t \geq 3\) the graph \(C_n^t\) has a super \((2n + 2, 1)\)-EAT labeling.

**Theorem D.**
For \(n \equiv 0 \pmod{4}\), \(n \geq 4\), and for \(t \equiv 2 \pmod{4}\), \(t \geq 2\) the graph \(C_n^t\) has a super \((2n + 2, 1)\)-EAT labeling.

However, the remaining cases are still open and Baca et. al \([1]\) conjectured the following.

**Conjecture 1.** There exists a super \((2n + 2, 1)\)-EAT labeling of \(C_n^t\) for

(i) \(n \equiv 0 \pmod{4}\) and \(t \equiv 0 \pmod{4}\), and
(ii) \(n \equiv 2 \pmod{4}\) and for \(t\) even.

For more results concerning antimagic total labeling, see for instances \([5, 8]\) and a nice survey paper by Gallian \([6]\).

The focus of this paper is on super EAT and VAT labelings of the Harary graph \(C_n^t\) and a disjoint union of \(k\) identical copies of \(C_n^t\).

## 2 On super \((a, d)\)-EAT labeling

In this section, we not only settle the above conjecture, but we also give a uniform construction for super edge-antimagic total labeling of \(C_n^t\) for any \(n \geq 4\) and any possible value of \(t \geq 2\). We also construct the super edge-antimagic total labeling for a disjoint union of \(k\) identical copies of the Harary graph.
Theorem 1. For any \( p \geq 4 \) and for any \( t \geq 2 \), \( G \cong C_p^t \) admits a super \((2p + 2, 1)\) edge-antimagic total labeling.

Proof. Let \( p = |V(G)| \) and \( q = |E(G)| \). We denote the vertex and edge sets of \( G \) are as follows:

\[
V = \{v_i : 1 \leq i \leq p\},
\]

\[
E = \{v_iv_{i+1} : 1 \leq i \leq p\} \cup \{v_iv_{i+t} : 1 \leq i \leq p\}.
\]

where all indices are taken in mod \( p \).

Now, we define labeling \( \lambda : V \cup E \rightarrow \{1, 2, \ldots, p + q\} \) as follows:

\[
\lambda(v_i) = i \quad \text{for} \quad 1 \leq i \leq p.
\]

\[
\lambda(v_iv_{i+1}) = \begin{cases} 
  p + q, & \text{for } i = p, \\
  p + q - i, & \text{for } 1 \leq i \leq p - 1.
\end{cases}
\]

\[
\lambda(v_iv_{i+t}) = q - \left\lfloor \frac{2i - 1}{2} \right\rfloor, \quad 1 \leq i \leq s.
\]

where \( s = \frac{p}{2} \) if \( p = 2t \) and \( s = p \) if \( p \neq 2t \).

We have two types of edges namely, \( E_1 = \{v_iv_{i+t} : 1 \leq i \leq p\} \) and \( E_2 = \{v_iv_{i+1} : 1 \leq i \leq p\} \). Now we consider all edge-weights. The edge-weights of all edges in \( E_1 \) will form consecutive integers \( 2p+2, 2p+3, \ldots, p+q+1 \). Note that the weight \( 2p+2 \) is obtained by the edge \( v_{p+1-t}v_1 \) if \( p \neq 2t \) or the edge \( v_1v_{1+t} \) if \( p = 2t \). Meanwhile, the edge-weights of all edges in \( E_2 \) will form consecutive integers \( p+q+2, p+q+3, \ldots, 2p+q+1 \). Therefore, all edge-weights form consecutive integers \( 2p+2, 2p+3, \ldots, 2p+q+1 \). Since all vertices receive the smallest labels then \( \lambda \) is a super \((2p+2, 1)\)-edge antimagic total labeling. \( \square \)

Figure 2 shows the super \((2p+2, 1)\)-EAT total labeling of \( C_8^4 \) and \( C_{10}^2 \).
By theorem A we have

**Corollary 1.** For any \( p \geq 4 \) and \( t \geq 2 \), \( G \cong C_p^t \) admits an \((a, 1)\) edge-antimagic total labeling, where \( a = 5p + 2 \) if \( p \neq 2t \) and \( a = 4p + 2 \) if \( p = 2t \).

Furthermore we show, in the following theorem that a disjoint union of \( k \) identical copies of the Harary graph admits a super edge-antimagic total labeling.

**Theorem 2.** For \( n \geq 4 \), \( k \geq 2 \) and \( t \geq 2 \), \( G \cong kC_n^t \) admits a super \((2nk + 2, 1)\) edge-antimagic total labeling.

**Proof.**
Let \( p = |V(G)| \) and \( q = |E(G)| \). We denote the vertex and edge sets of \( G \) as follows.

\[
V = \{ v_i^j : 1 \leq i \leq n, 1 \leq j \leq k \},
\]

\[
E = \{ v_i^j v_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq k \} \cup \{ v_i^j v_{i+t}^j : 1 \leq i \leq n, 1 \leq j \leq k \}.
\]

where all indices are taken in mod \( n \).

Now, we define labeling \( \lambda : V \cup E \rightarrow \{1, 2, ..., p + q\} \) as follows:

\[
\lambda(v_i^j) = (i - 1)k + j \quad 1 \leq i \leq n, 1 \leq j \leq k.
\]

\[
\lambda(v_i^j v_{i+1}^j) = \begin{cases} 
    p + q - (j - 1), & i = n, 1 \leq j \leq k \\
    p + q - ki - (j - 1), & 1 \leq i \leq n - 1, 1 \leq j \leq k.
\end{cases}
\]
\[ \lambda(v^j_i v^j_{i+t}) = q - k\left[\frac{2i - 1}{2}\right] - (j - 1), \quad 1 \leq i \leq s, \quad 1 \leq j \leq k. \]

where \( s = \frac{n}{2} \) if \( n = 2t \) and \( s = n \) if \( n \neq 2t \).

We have two types of edges namely, \( E_1 = \{v^j_i v^j_{i+t} : 1 \leq i \leq n, \ 1 \leq j \leq k\} \) and \( E_2 = \{v^j_i v^j_{i+1} : 1 \leq i \leq n, \ 1 \leq j \leq k\} \). Now we consider all edge-weights. The edge-weights of all edges in \( E_1 \) will form consecutive integers \( 2nk + 2, 2nk + 3, ..., 3nk + 1 \). Note that the weight \( 2nk + 2 \) is obtained by the edge \( v^j_{n+1-t}v^1_1 \) if \( n \neq 2t \) or the edge \( v^j_1v^1_{1+t} \) if \( n = 2t \). Meanwhile the edge-weights of all edges in \( E_2 \) will form consecutive integers \( 3nk + 2, 3nk + 3, ..., 4nk + 1 \). Therefore, all edge-weights form consecutive integers \( 2nk + 2, 2nk + 3, ..., 4nk + 1 \). Since all vertices receive the smallest labels then \( \lambda \) is a super \((2nk + 2,1)\)-edge antimagic total labeling.

Figure 3 shows the super \((2nk+2,1)\)-EAT total labeling of \( 2C_8^3 \).

\[ \text{Fig. 3. The super (34,1)-EAT labeling of } 2C_8^3. \]

By theorem A we have

**Corollary 2.** For any \( n \geq 4 \) and for any \( t \geq 2 \), \( G \cong kC_n^t \) admits an \((a,1)\) edge-antimagic total labeling, where \( a = 5nk + 2 \) if \( n \neq 2t \) and \( a = 4nk + 2 \) if \( n = 2t \).
3 On super \((a, d)\)-VAT labeling

In this section we construct \((a, d)\)-VAT labeling of Harary graph \(C^t_n\) as well as for disjoint union of \(k\) identical copies of Harary graphs. Before proving our main results let us prove the following fact:

**Lemma 1.** Let \(t \geq 2\) and \(p \geq 5\). If Harary graph \(G \cong C^t_p\) is super \((a, d)\) vertex-antimagic total then \(d < 9\) for \(p \neq 2t\) and \(d < 6\) for \(p = 2t\).

**Proof.** Let \(p = |V(G)|\) and \(q = |E(G)|\). Assume that there exists a bijection

\[ \lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., p + q\} \]

which is super \((a, d)\)-vertex antimagic total and

\[ W = \{\lambda(u) + \sum \lambda(uv) : uv \in E(G)\} = \{a, a + d, ..., a + (p - 1)d\} \]

is the set of vertex-weights.

If \(p \neq 2t\), then minimum possible vertex-weight in a super \((a, d)\)-vertex antimagic total labeling is

\[ 1 + (p + 1) + (p + 2) + (p + 3) + (p + 4) = 4p + 11. \]

and maximum vertex-weight is no more than

\[ p + 3p + (3p - 1) + (3p - 2) + (3p - 3) = 13p - 6. \]

Thus, we have

\[ a + (p - 1)d \leq 13p - 6, \]

and

\[ d \leq \frac{9p - 17}{p - 1} < 9. \]

If \(p = 2t\), then minimum possible vertex-weight in a super \((a, d)\)-vertex antimagic total labeling is

\[ 1 + (p + 1) + (p + 2) + (p + 3) = 3p + 7. \]

and maximum vertex-weight is no more than

\[ p + \frac{5p}{2} + \left(\frac{5p}{2} - 1\right) + \left(\frac{5p}{2} - 2\right) = \frac{17p}{2} - 3. \]
Thus, we have
\[ a + (p - 1)d \leq \frac{17p}{2} - 3, \]
and
\[ d \leq \frac{11p - 20}{2(p - 1)} < 6. \quad \square \]

**Theorem 3.** For \( p \geq 5 \) and \( t \geq 2 \), \( G \cong C_p^t \) admits a super \((8p+3, 1)\) vertex-antimagic total labeling, provided if \( p \neq 2t \).

**Proof.**
Let \( p = |V(G)| \) and \( q = |E(G)| \), then we denote the vertex and edge sets of \( G \) as follows:
\[
V = \{v_i : 1 \leq i \leq p \},
E = \{v_iv_{i+1} : 1 \leq i \leq p \} \cup \{v_iv_{i+t} : 1 \leq i \leq p \}.
\]
where all indices are taken in mod \( p \).

Now, we define the labeling \( \lambda : V \cup E \to \{1, 2, ..., p + q\} \) as follows:
\[
\lambda(v_i) = i \quad 1 \leq i \leq p.
\]
\[
\lambda(v_iv_{i+1}) = \begin{cases} 
2p, & \text{for } i = p, \\
2p - i, & \text{for } 1 \leq i \leq p - 1.
\end{cases}
\]
\[
\lambda(v_iv_{i+t}) = \begin{cases} 
2p + 1, & \text{for } i = p, \\
2p + 2 + \left\lfloor \frac{2t-1}{2} \right\rfloor, & \text{for } 1 \leq i \leq p - 1.
\end{cases}
\]
We have vertices namely, \( \{v_i : 1 \leq i \leq p\} \) and we can see that the vertex \( v_t \) has the weight \( 8p + 3 \) and set of vertices form a consecutive sequence \( 8p+3, 8p+4, ..., 9p+2 \) with \( a = 8p+3 \) and \( d = 1 \). Since all vertices receive the smallest labels then \( \lambda \) is a super \((8p+3, 1)\)-vertex antimagic total labeling. \quad \square

**Theorem 4.** For \( n \geq 5, k \geq 2 \) and \( t \geq 2 \), \( G \cong kC_n^t \) admits a super \((8nk + 3, 1)\) vertex-antimagic total labeling, provided if \( p \neq 2t \).

**Proof.**
Let \( p = |V(G)| \) and \( q = |E(G)| \) then
\[
p = nk,
q = 2nk,
\]
and

\[ V = \{v^j_i : 1 \leq i \leq n, 1 \leq j \leq k\}, \]
\[ E = \{v^j_i v^j_{i+1} : 1 \leq i \leq n, 1 \leq j \leq k\} \cup \{v^j_i v^j_{i+t} : 1 \leq i \leq n, 1 \leq j \leq k\} \]

where all indices are taken in mod \( n \).

Now, we define labeling \( \lambda : V \cup E \to \{1, 2, \ldots, p + q\} \) as follows:

\[
\lambda(v^j_i) = (i - 1)k + j \quad 1 \leq i \leq n, \ 1 \leq j \leq k.
\]
\[
\lambda(v^j_i v^j_{i+1}) = \begin{cases} 
2p - (j - 1), & \text{for } i = n, \ 1 \leq j \leq k \\
2p + 1 - (j + ki), & \text{for } 1 \leq i \leq n - 1, \ 1 \leq j \leq k.
\end{cases}
\]
\[
\lambda(v^j_i v^j_{i+t}) = \begin{cases} 
2p + j, & \text{for } i = n, \ 1 \leq j \leq k \\
2p + ik + j, & \text{for } 1 \leq i \leq n - 1, \ 1 \leq j \leq k.
\end{cases}
\]

We have vertices namely, \( \{v^j_i : 1 \leq i \leq n, 1 \leq j \leq k\} \) and we can see that the vertex \( v^1_i \) has the weight \( 8nk + 3 \) and set of vertices form a consecutive sequence \( 8nk + 3, 8nk + 4, \ldots, 9nk + 2 \) with \( a = 8nk + 3 \) and \( d = 1 \). Since all vertices receive the smallest labels then \( \lambda \) is a super \((7nk + 3, 1)\)-vertex antimagic total labeling.

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References